On some q-identities related to divisor functions

Jiang Zeng

Institut Girard Desargues, Université Claude Bernard (Lyon I)
21 Avenue Claude Bernard, 69622 Villeurbanne Cedex, France
e-mail: zeng@igd.univ-lyon1.fr

Abstract: We give generalizations and simple proofs of some q-identities of Dilcher, Fu and Lascoux related to divisor functions.

Let $a_1, \ldots, a_N$ be N indeterminates. It is easy to see that

$$\frac{1}{(1-a_1z)(1-a_2z)\ldots(1-a_Nz)} = \sum_{k=1}^{N} \frac{\prod_{j=1,j\neq k}^{N}(1-a_j/a_k)^{-1}}{1-a_kz}. \tag{1}$$

The coefficient of z^τ ($\tau \geq 0$) in the left side of (1) is usually called the τ-th *complete symmetric function* $h_\tau(a_1, \ldots, a_N)$ of $a_1, \ldots, a_N$. Clearly, we have $h_0(a_1, \ldots, a_N) = 1$ and equating the coefficients of z^τ ($\tau \geq 1$) in two sides of (1) yields

$$h_\tau(a_1, \ldots, a_N) := \sum_{1\leq i_1\leq i_2 \leq \cdots \leq i_\tau \leq N} a_{i_1}a_{i_2}\ldots a_{i_\tau} = \sum_{k=1}^{N} \prod_{j=1,j\neq k}^{N} (1-a_j/a_k)^{-1}\, a_k^\tau. \tag{2}$$

In particular, if $a_k = \frac{a-bq^{k+i-1}}{c-zq^{k+i-1}}$ ($1 \leq k \leq N$) for a fixed integer i ($1 \leq i \leq n$), then formula (2) with $N = n-i+1$ reads

$$\begin{aligned} h_\tau\left(\frac{a-bq^i}{c-zq^i}, \frac{a-bq^{i+1}}{c-zq^{i+1}}, \ldots, \frac{a-bq^n}{c-zq^n}\right) &= \frac{c^{n-i+1}(zq^i/c)_{n-i+1}}{(q)_{n-i+1}(az-bc)^{n-i}} \\ \cdot \sum_{k=i}^{n} (-1)^{k-i} \begin{bmatrix} n-i+1 \\ n-k \end{bmatrix} q^{\binom{k-i+1}{2}-k(n-i)} &\frac{(1-q^{k-i+1})(a-bq^k)^{\tau+n-i}}{(c-zq^k)^{\tau+1}}, \end{aligned} \tag{3}$$

where $(x)_n = (1-x)(1-xq)\ldots(1-xq^{n-1})$ and $\begin{bmatrix} n \\ i \end{bmatrix} = (q^{n-i+1})_i/(q)_i$ with $(x)_0 = 1$.

The aim of this note is to show that (3) turns out to be a common source of several q-identities surfacing recently in the literature.

First of all, the $i = 1$ case of formula (3) with $\tau = m-n+1$ corresponds to an identity of Fu and Lascoux [3, Prop. 2.1]:

$$\begin{aligned} &h_\tau\left(\frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \ldots, \frac{a-bq^n}{c-zq^n}\right) \\ &= \frac{c^n(zq/c)_n}{(q)_n(az-bc)^{n-1}} \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{k-1} q^{\binom{k+1}{2}-nk} \frac{(1-q^k)(a-bq^k)^m}{(c-zq^k)^{\tau+1}}. \end{aligned} \tag{4}$$

Next, for $i = 1, \ldots, n$ and $m \geq 1$ set

$$A_i(z) := \frac{q^i (zq)_{i-1} (q)_n}{(q)_i (zq)_n} h_{m-1} \left(\frac{q^i}{1 - zq^i}, \ldots, \frac{q^n}{1 - zq^n} \right). \tag{5}$$

Then we have the following polynomial identity in x:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1)\cdots(x-q^{k-1})}{(1-zq^k)^m} q^{mk} = \sum_{k=1}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+mk}}{(1-zq^k)^m} + \sum_{i=1}^{n} A_i(z) x^i. \tag{6}$$

Indeed, using the q-binomial formula [1, p. 36]:

$$(x-1)(x-q)\ldots(x-q^{N-1}) = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^{N-j} x^j q^{\binom{N-j}{2}},$$

we see that the coefficient of x^i $(1 \leq i \leq n)$ in the left side of (6) is equal to

$$\sum_{k=i}^{n} (-1)^{k-i} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \frac{q^{mk+\binom{k-i}{2}}}{(1-zq^k)^m} = \frac{q^i (zq)_{i-1} (q)_n}{(q)_i (zq)_n} h_{m-1} \left(\frac{q^i}{1-zq^i}, \ldots, \frac{q^n}{1-zq^n} \right), \tag{7}$$

where the last equality follows from (3) with $a = 0$, $c = 1$, $b = -1$ and $\tau = m - 1$.

Now, with $z = i = 1$ and m shifted to $m + 1$, formula (7) reduces to Dilcher's identity [2]:

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1} q^{\binom{k}{2}+mk}}{(1-q^k)^m} = h_m \left(\frac{q}{1-q}, \ldots, \frac{q^n}{1-q^n} \right) = \sum_{i=1}^{n} A_i(1).$$

On the other hand, formula (1) with $N = n + 1$ and $a_i = q^{i-1}$ $(1 \leq i \leq N)$ yields

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+k}}{1-zq^k} = \frac{(q)_n}{(z)_{n+1}}.$$

Hence, setting, respectively, $z = 1$ and $m = 1$ in formula (6) we recover two recent formulae of Fu and Lascoux [4] (see also [5]):

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1)\cdots(x-q^{k-1})}{(1-q^k)^m} q^{mk} = \sum_{i=1}^{n} (x^i - 1) A_i(1), \tag{8}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1)\cdots(x-q^{k-1})}{1-zq^k} q^k = \frac{(q)_n}{(z)_{n+1}} \sum_{i=0}^{n} \frac{(z)_i}{(q)_i} x^i q^i. \tag{9}$$

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